Announcements

1) Turfe Lecture – general audience math talk, Wednesday next week, 3-4, CB 1030

Stephen DeBacker:

connections between geometry and number theory

2) PIC BIG – research opportunity for undergrads, send e-mail if interested

Extra Credit

(10 points, due 4/1)

Show that the inner product on $\mathbb{R}^2$ (or $\mathbb{C}^2$, if you like) is backwards stable.

Turn in individual solutions.

The Problem:

(Leture 16)

Given $A \in \mathbb{C}^{m \times n}$, compute its QR factorization via an algorithm. How close is the computed product to A?

Depends on the algorithm, we'll use 10.1 (Householder reflection method)

Householder Experiment

(p. 114)

Find a matrix with determined QR decomposition, compare what happens when we use Matlab's "qr" command. We see the computed $\tilde{Q}$ and $\tilde{R}$ aren't that close to Q and R, but $\tilde{Q}\tilde{R}$ is really close to A.

Strong Statement of Problem:

$$f: \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times m} \times \mathbb{C}^{m\times n}$$

$$f(A) = (Q, R)$$

where

1) Q is unitary
2) R is upper triangular
3) $A = QR$.

Algorithm:

$$\tilde{f} : \mathbb{C}^{m \times n} \to \mathbb{C}^{m \times m} \times \mathbb{C}^{m \times n}$$

where

$$\tilde{f}(A) = (\tilde{Q}, \tilde{R})$$

where we need to specify $\tilde{Q}$ and $\tilde{R}$.

What are $\tilde{Q}$ and $\tilde{R}$?

$\boxed{\tilde{R} = \text{upper triangular matrix obtained using algorithm 10.1 on } A.}$

$\tilde{Q}$ is a bit more problematic.

v_k = vectors produced by algorithm 10.1 (determine Q)

$\tilde{v}_k$ = floating point approximations to v_k

$\tilde{Q}_k$ = Householder reflections corresponding to $\tilde{v}_k$

(these are all unitary)

$$\boxed{\tilde{Q} = \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3 \cdots \tilde{Q}_n}$$

($\tilde{Q}$ is unitary)

Theorem: (backwards stability of Householder)

Given $A \in \mathbb{C}^{m\times n}$, the factors $\tilde{Q}$ and $\tilde{R}$ constructed using algorithm 10.1 satisfy

$$\boxed{\tilde{Q}\tilde{R} = A + \delta A = \tilde{A}}$$

where $\dfrac{\|\delta A\|}{\|A\|} = O(\varepsilon_{machine})$

Algorithm for solving $Ax = b$ by QR Factorization

Steps:

1) Find the QR decomposition of A via algorithm 10.1 (backwards stable)

2) Compute the vector $Q^* b$. (backwards stable?)

3) Solve for x by solving $Rx = Q^* b$ via back-substitution (backwards stable?)

In fact, 2) is backwards stable (a bunch of dot products, all backwards stable).

We need an algorithm for 3) that is backwards stable. we use the natural one.

Example 1: (Matlab, then by hand)

$$A = \begin{bmatrix} 3/5 & -6/5 \\ 4/5 & 17/5 \end{bmatrix}$$

$A = QR$ where

$$Q = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.$$

Solve

$$Ax = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

via QR decomposition:

$$R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q^* \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 29 \\ -22 \end{bmatrix}$$

$$R\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 \end{bmatrix}$$

Should equal

$$\frac{1}{5}\begin{bmatrix} 29 \\ -22 \end{bmatrix}.$$

So $3x_2 = -\frac{22}{5}$,

$$\boxed{x_2 = -\frac{22}{15}}$$

$$x_1 + 2x_2 = \frac{29}{5}$$

$$x_1 = \frac{29}{5} - 2x_2$$

$$= \frac{29}{5} + \frac{44}{15}$$

$$= \boxed{\frac{131}{15}}$$